Indian Statistical Institute
Bangalore Centre
B.Math (Hons.) III Year 2016-2017
First Semester
Statistics III

Back paper Examination — Date : 29.12.16

Time : 3 hours

Answer as many questions as possible. The maximum you can score is 100.
All symbols have their usual meaning, unless stated otherwise.
State clearly the results you use.

1. Suppose $X_1, \cdots X_n$ are independent normal variables with mean $\mu_1, \cdots \mu_n$ and common variance σ^2. Further, suppose X_0 also follows normal distribution with mean 0, variance σ^2 and $Corr(X_0, X_i) = \rho, 1 \leq i \leq n$. Define $Y_i = X_i - aX_0, 1 \leq i \leq n$. Show that $(Y_1, \cdots Y_n)'$ follows multivariate normal with covariance matrix of the form $aI_n + bJ_n$. [Here J_n is the $n \times n$ all-one matrix]. [10]

2. Consider the linear regression model $y_i = \alpha + \beta x_i + \epsilon_i, \; 1 \leq i \leq n$. Here α, β are unknown constants, X_i's are known constants and $\epsilon_i, i = 1, 2, \cdots n$ are i.i.d normally distributed variables with mean 0 and variance σ^2. Suppose

$$\begin{aligned} S_x^2 &= \sum_{i=1}^{n}(x_i - \bar{x})^2, \\ S_y^2 &= \sum_{i=1}^{n}(y_i - \bar{y})^2 \text{ and} \\ S_{xy} &= \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}). \end{aligned}$$

(a) Show that $E(S_{xy}^2) = \sigma^2 S_x^2 + \beta^2 S_x^4$.

(b) Write down the expression for the error sum of squares (S_E^2). Derive the distribution of S_E^2 without using any result of least square theory. [6 + (1 + 7) = 14]

3. Consider the linear model

$$Y(n\times1) = X(n\times p)\beta(p\times1)+\varepsilon(n\times1), \quad \text{where } E(\varepsilon) = 0, \; Cov(\varepsilon) = \sigma^2 I_n.$$

(a) Suppose l is in R^p. When is $l'\beta$ said to be estimable?

(b) How does one find a least square estimate ($\hat{\beta}$) of β ? Is it always unique ?

(c) Suppose $l'\beta$ is estimable. Show that

(i) $l'\hat{\beta}$ is unbiased for $l'\beta$,

(ii) $l'\hat{\beta}$ is unique and

(ii) it has minimum variance among all linear unbiased estimators of $l'\beta$.

$[1 + (2 + 1) + (3 + 4 + 5) = 16]$

4. Consider the model

$$y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij},\ j = 1, 2, ..., b,\ i = 1, 2...a$$

(a) For each of the following parametric functions, either provide an unbiased estimator or prove that it is not estimable. Assume $b \geq 3$.

(i) μ, (ii) β_1, (iii) $7\alpha_1 - 5\alpha_2 - 2\alpha_3$.

(b) Write down the model in the form $Y = X_0\ \mu + X_1\ \alpha + X_2\ \beta + \varepsilon$, describing the matrices X_0, X_1, X_2

(c) Show that the rank of $X = [X_0|X_1|X_2]$ is $a + b - 1$

$[(3 \text{ x } 3) + 3 + 5 = 17]$

5. Suppose X follows $N_p(0, I_p)$ and A is a $p \times p$ symmetric matrix of rank r.

(a) Prove that a necessary and sufficient condition for $X'AX$ to follow $\chi^2(r)$ is that A is idempotent.

(b) Suppose $l \in R^P$. Show that if $Al = 0$, then $l'X$ and $X'AX$ are independent. $[7 + 9 = 16]$

6. (a) Define Wishart distribution. When is it called central ?

(b) Suppose $X_i,\ i = 1, 2, \cdots n$ are i.i.d. $N_p(0, \Sigma)$ and A is a $p \times p$ symmetric matrix.

Let $X(n \times p) = \begin{bmatrix} X_1' \\ X_2' \\ \vdots \\ X_n' \end{bmatrix}.$

Prove that a necessary and sufficient condition for $S = X'AX$ to follow $W_p(n, \Sigma)$ is that $l'Sl/l'\Sigma l \sim \chi^2(r)$ for every $l \in R^p$.

$[2 + 10 = 12]$

7. Suppose $X_i,\ i = 1, 2, \cdots n$ are i.i.d. $N_p(\mu, \Sigma)$. Define $S^2 = \sum_{i=1}^n (X_i - \bar{X})(X_i - \bar{X})'$. Show that

(a) $\bar{X}$ and S^2 are independent and

(b) $S^2 \sim W_p(n - 1, \Sigma)$.

$[10 + 5 = 15]$